LIMITS, CONTINUITY & DIFFERENTIABILITY — TIPS, TRICKS, TRAPS & APPLICATIONS

A1. FUNCTIONS & ALGEBRA OF FUNCTIONS

- Domain of $f \pm g$, $fg = D_f \cap D_g$; of $\frac{f}{g} = D_f \cap D_g \setminus \{g(x) = 0\}$.
- $f \circ g$: need $x \in D_g$ *and* $g(x) \in D_f$. Generally $f \circ g \neq g \circ f$.
- Domain musts: $\sqrt{u} \Rightarrow u \geq 0$; $\ln u \Rightarrow u > 0$; $\log_u v \Rightarrow u > 0, u \neq 1, v > 0$; $\sin^{-1} u, \cos^{-1} u \Rightarrow |u| \leq 1$; $\frac{1}{u} \Rightarrow u \neq 0$.
- Even: $f(-x) = f(x)$ (symmetric about y-axis); odd: $f(-x) = -f(x)$. Every $f = \frac{f(x)+f(-x)}{2} + \frac{f(x)-f(-x)}{2}$.
- Period: $\sin(ax+b) \to \frac{2\pi}{|a|}$, $\tan, |\sin|, |\cos| \to \pi$, $\{x\} \to 1$, $|\sin x| + |\cos x| \to \frac{\pi}{2}$. Sum: LCM of periods (check for smaller).
- **Trap:** Simplifying before finding domain: $f(x) = \frac{x^2-1}{x-1}$ is *not* $x+1$ at $x = 1$; $\ln x^2 \neq 2\ln x$ for $x < 0$.
- **Trap:** $\sin^4 x + \cos^4 x$ has period $\frac{\pi}{2}$, not π or 2π — always test a smaller candidate.

A2. STANDARD FUNCTIONS & GRAPHS

- **GIF** $[x]$: $x - 1 < [x] \leq x$; $[x+n] = [x] + n$; $[x] + [-x] = 0$ if $x \in \mathbb{Z}$, else -1.
- **Fractional part** $\{x\} = x - [x] \in [0, 1)$; **sgn** $x = \frac{|x|}{x}$ $(x \neq 0)$, 0 at 0.
- a^x: increasing if $a > 1$, decreasing if $0 < a < 1$; $\log_a x$ is its mirror in $y = x$.
- Rational $\frac{P}{Q}$: vertical asymptote where $Q = 0$ (after cancelling); horizontal from degree comparison.
- **Transformations:** $f(x+a)$ shift left a; $f(x) + a$ up; $f(-x)$ reflect in y-axis; $|f(x)|$ flip negative part up; $f(|x|)$ keep $x \geq 0$ part, mirror it; $f(ax)$ compress by a.
- **Trick:** Number of solutions of $f(x) = g(x)$ = intersections of graphs — sketch, don't solve (e.g. $\sin x = \frac{x}{10}$, $|x| = e^{-x}$).
- **Trap:** $[-x] \neq -[x]$ for non-integers: $[-2.3] = -3$.

A3. INVERSE FUNCTIONS & INVERSE TRIG

- f^{-1} exists iff f is **bijective**. Graph of f^{-1} = mirror of f in $y = x$. Find: write $y = f(x)$, solve for x, swap.
- If f is increasing, solutions of $f(x) = f^{-1}(x)$ lie on $y = x$: just solve $f(x) = x$.
- Principal ranges: $\sin^{-1} : [-\frac{\pi}{2}, \frac{\pi}{2}]$, $\cos^{-1} : [0, \pi]$, $\tan^{-1} : (-\frac{\pi}{2}, \frac{\pi}{2})$, $\cot^{-1} : (0, \pi)$, $\sec^{-1} : [0, \pi] \setminus \{\frac{\pi}{2}\}$, $\csc^{-1} : [-\frac{\pi}{2}, \frac{\pi}{2}] \setminus \{0\}$.
- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$; $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$; $\sin^{-1}(-x) = -\sin^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1} x$.
- **Trap:** $\sin^{-1}(\sin x) = x$ only for $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$: $\sin^{-1}(\sin 2) = \pi - 2$. But $\sin(\sin^{-1} x) = x$ always on $[-1, 1]$.

A4. STANDARD LIMITS (memorise cold)

$$\lim_{x\to0}\frac{\sin x}{x} = \lim_{x\to0}\frac{\tan x}{x} = \lim_{x\to0}\frac{\sin^{-1}x}{x} = \lim_{x\to0}\frac{\tan^{-1}x}{x} = 1$$

$$\lim_{x\to0}\frac{1-\cos x}{x^2} = \frac{1}{2};\quad \lim_{x\to0}\frac{e^x-1}{x} = 1;\quad \lim_{x\to0}\frac{a^x-1}{x} = \ln a$$

$$\lim_{x\to0}\frac{\ln(1+x)}{x} = 1;\quad \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1};\quad \lim_{x\to0}\frac{(1+x)^n-1}{x} = n$$

$$\lim_{x\to0}(1+x)^{1/x} = e;\quad \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^x = e^a$$

1^∞ **form:** $\lim f^g = e^{\lim g\,(f-1)}$ — the single most tested limit rule.

- **Series (first terms):** $\sin x = x - \frac{x^3}{6}$, $\tan x = x + \frac{x^3}{3}$, $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24}$, $e^x = 1 + x + \frac{x^2}{2}$, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3}$, $(1+x)^n = 1 + nx + \frac{n(n-1)}{2}x^2$.
- $\lim_{x\to0}\frac{x-\sin x}{x^3} = \frac{1}{6}$, $\frac{\tan x - x}{x^3} = \frac{1}{3}$, $\frac{\tan x - \sin x}{x^3} = \frac{1}{2}$.

A5. LIMIT TECHNIQUES & TRAPS

- **Indeterminate:** $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty - \infty, 1^\infty, 0^0, \infty^0$. Anything else, just substitute.
- $\frac{0}{0}$ algebraic: factorise / rationalise (multiply by conjugate). $\frac{\infty}{\infty}$ rational at ∞: divide by highest power ($\deg P < \deg Q \Rightarrow 0$; equal $\Rightarrow$ ratio of leading coeffs).
- **L'Hôpital** for $\frac{0}{0}, \frac{\infty}{\infty}$; but series expansion is faster for x^3-type denominators and gives unknown constants directly.
- **Sandwich:** $g \leq f \leq h$, $\lim g = \lim h \Rightarrow \lim f$ same. Used for $[x]$-sums, $\frac{\sin x}{x}$ at ∞ $(= 0)$.
- "Find a, b so limit is finite": expand in series, force the coefficients of negative powers to vanish.
- lim exists $\iff$ LHL = RHL (finite). Check both sides for $[x], |x|, \{x\}, e^{1/x}, \tan^{-1}\frac{1}{x}$ at the critical point.
- **Trap:** Replacing $\sin x$ by x inside a *difference*: $\frac{x-\sin x}{x^3} \neq 0$. Equivalents are safe only in products/quotients.
- **Trap:** $\sqrt{x^2} = |x|$: as $x \to -\infty$, $\sqrt{x^2+x} = -x\sqrt{1+\frac{1}{x}}$. Sign errors here are the top loss.
- **Trap:** $\lim_{x\to0} e^{1/x}$ does not exist (RHL $= \infty$, LHL $= 0$); $\lim_{x\to0}[x]$ does not exist; $\lim_{x\to0}[x^2] = 0$.
- **Growth order at ∞:** $\ln x \ll x^n \ll a^x$ $(a > 1) \ll x! \ll x^x$; so $\frac{x^n}{e^x} \to 0$, $\frac{\ln x}{x^n} \to 0$; at 0^+: $x \ln x \to 0$, $x^x \to 1$.
- **Trick:** $\lim_{x\to\infty}\left(\frac{x+a}{x+b}\right)^x = e^{a-b}$; $\lim_{n\to\infty}\left(1+\frac{a}{n}+\frac{b}{n^2}\right)^n = e^a$ — direct 1^∞ shortcuts.

B1. CONTINUITY

f continuous at $a \iff \lim_{x\to a^-} f = \lim_{x\to a^+} f = f(a)$ (all three exist and are equal).

- Sums, products, quotients $(g \neq 0)$, composites of continuous functions are continuous. Polynomials, $\sin, \cos, e^x$ continuous everywhere; $\ln, \tan$ on their domains.
- **Removable:** limit exists $\neq f(a)$ (or $f(a)$ undefined). **Jump:** LHL $\neq$ RHL. **Infinite/oscillatory:** $\frac{1}{x}$, $\sin\frac{1}{x}$ at 0.
- $[g(x)]$ can break only where $g(x)$ is an integer — check those points only. $[x]$ is right-continuous at integers.
- Continuous f on $[a, b]$: attains max & min, and takes every value between $f(a), f(b)$ (**IVT**) — proves a root exists.
- **Trick:** Continuous + non-continuous may be continuous; continuous + discontinuous is *always* discontinuous. Product of two discontinuous functions can be continuous.
- **Trap:** Piecewise f with parameters: equate LHL, RHL *and* $f(a)$. Forgetting $f(a)$ loses "removable" answers.

B2. DIFFERENTIABILITY

$f'(a)$ exists $\iff$ LHD $= \lim_{h\to0^-}\frac{f(a+h)-f(a)}{h} =$ RHD (finite).

- Differentiable $\Rightarrow$ continuous; **not** conversely ($|x|$ at 0). Discontinuous $\Rightarrow$ not differentiable — check continuity first.
- Non-differentiable at: corners (LHD $\neq$ RHD), vertical tangents ($x^{1/3}$ at 0), discontinuities.
- $|f(x)|$ is non-differentiable where f has a root of **odd** multiplicity with $f' \neq 0$; $|x^3|$, $|x-1|^2$ *are* differentiable.
- $|x-a|\,g(x)$ differentiable at $a \iff g(a) = 0$. So $x|x|$ is differentiable at 0.
- $x^n \sin\frac{1}{x}$ $(f(0) = 0)$: $n = 1$ continuous only; $n = 2$ differentiable but f' discontinuous; $n \geq 3$ f' continuous.
- max / min(f, g): candidates are crossing points where slopes differ — sketch and count corners.
- **Trap:** Differentiating by formula and then checking $f'(a)$ at a piecewise joint. Use the *definition* (first-principles) at joints.

B3. DERIVATIVE RULES & TABLE

$(uv)' = u'v + uv'$; $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$; chain: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$.

$\sin x$	$\to \cos x$	$\sin^{-1} x$	$\to \frac{1}{\sqrt{1-x^2}}$
$\cos x$	$\to -\sin x$	$\cos^{-1} x$	$\to \frac{-1}{\sqrt{1-x^2}}$
$\tan x$	$\to \sec^2 x$	$\tan^{-1} x$	$\to \frac{1}{1+x^2}$
$\cot x$	$\to -\csc^2 x$	$\cot^{-1} x$	$\to \frac{-1}{1+x^2}$
$\sec x$	$\to \sec x \tan x$	$\sec^{-1} x$	$\to \frac{1}{\lvert x\rvert\sqrt{x^2-1}}$
$\csc x$	$\to -\csc x \cot x$	$\csc^{-1} x$	$\to \frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$
a^x	$\to a^x \ln a$	$\log_a x$	$\to \frac{1}{x \ln a}$

- **Trick:** Simplify inverse-trig first by substitution: $x = \tan\theta$ gives $\sin^{-1}\frac{2x}{1+x^2} = 2\tan^{-1} x$ for $|x| \leq 1$, $\tan^{-1}\frac{2x}{1-x^2} = 2\tan^{-1} x$ for $|x| < 1$.
- **Trap:** Those identities change outside the stated range (e.g. $\pi - 2\tan^{-1} x$ for $x > 1$) — the derivative's sign flips.
- **Trap:** $\frac{d}{dx}\sin(x^\circ) = \frac{\pi}{180}\cos(x^\circ)$ — degrees carry a factor.

B4. LOG, IMPLICIT, INVERSE & 2ND ORDER

- **Log diff.** for f^g or long products: $y = x^x \Rightarrow y' = x^x(1 + \ln x)$; $\frac{d}{dx}f^g = f^g\left(g'\ln f + \frac{gf'}{f}\right)$.
- **Implicit:** differentiate both sides, collect y'; for $F(x, y) = 0$, $\frac{dy}{dx} = -\frac{F_x}{F_y}$.
- $y = \sqrt{f + \sqrt{f + \cdots}}$: $y^2 = f + y \Rightarrow y' = \frac{f'}{2y-1}$.
- **Parametric:** $\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$; $\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\cdot\frac{dt}{dx}$.
- **Inverse:** $(f^{-1})'(f(a)) = \frac{1}{f'(a)}$; $\frac{d^2x}{dy^2} = -\frac{d^2y/dx^2}{(dy/dx)^3}$.
- **Trap:** $\frac{d^2y}{dx^2} \neq \frac{d^2y/dt^2}{d^2x/dt^2}$ and $\frac{d^2x}{dy^2} \neq \frac{1}{d^2y/dx^2}$.

C1. RATE OF CHANGE

- $\frac{dy}{dt} = \frac{dy}{dx}\cdot\frac{dx}{dt}$. Write the geometric relation *first*, differentiate in t, substitute values *last*.
- Sphere: $\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$, $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$; cone with fixed $\frac{h}{r}$: eliminate one variable before differentiating.
- **Trap:** Substituting the instantaneous value before differentiating (turns a variable into a constant). Decreasing quantity $\Rightarrow$ negative rate.

C2. INCREASING & DECREASING

- $f' > 0$ on $I \Rightarrow$ strictly increasing; $f' \geq 0$ with $f' = 0$ only at isolated points still strictly increasing (x^3).
- Sign chart of f' using its zeros *and* points where f' or f is undefined.
- **Trick:** Compare numbers via monotonicity: $\frac{\ln x}{x}$ peaks at $x = e$ $\Rightarrow e^\pi > \pi^e$; prove inequalities by $g = f_1 - f_2$, $g(0) = 0$, $g' > 0$.
- **Trap:** $f' > 0$ on each piece of a broken domain $\neq$ increasing overall: $-\frac{1}{x}$ has $f' > 0$ but $f(-1) > f(1)$.

C3. MAXIMA & MINIMA

- Critical points: $f' = 0$ *or* f' undefined. First-derivative test: $+ \to -$ max, $- \to +$ min.
- 2nd test: $f'(c) = 0$, $f''(c) < 0$ max, > 0 min; $f''(c) = 0 \Rightarrow$ inconclusive (use first test / higher derivative: first non-zero even order decides).
- **Global on** $[a, b]$**:** compare f at critical points *and endpoints*.
- Classics: fixed perimeter $\Rightarrow$ square max area; rectangle in circle $\Rightarrow$ square; max of $a\sin x + b\cos x = \sqrt{a^2+b^2}$; AM–GM for products with fixed sum.
- **Trap:** Local max can be smaller than a local min. Corners ($|x|$ at 0) are extrema with no $f' = 0$. x^3: $f'(0) = 0$ yet no extremum.

EXAM-DAY CHECKLIST

- Identify the indeterminate form before choosing a method.
- For piecewise/mod/GIF questions, list the breakpoints first and test only those.
- For MCQs on limits, substitute a small number (e.g. $x = 0.01$) to sanity-check the option.
- Always check endpoints and undefined points in monotonicity and extremum questions.

For personal revision only