DIFFERENTIAL EQUATIONS: ORDER, DEGREE & SOLUTIONS — TIPS, TRICKS & TRAPS

A1. ODE BASICS

- **ODE:** relation between x, y and derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots$ of one independent variable.
- **General solution** has as many arbitrary constants as the **order**; **particular solution** fixes them from given conditions.
- **Verifying a solution:** differentiate the candidate and substitute — often faster than solving in MCQs.
- **Trap:** A "solution" with fewer constants than the order is not the general solution.

A2. ORDER & DEGREE

- **Order** = order of the highest derivative present.
- **Degree** = power of the highest-order derivative *after* the DE is made a polynomial in all derivatives (clear radicals and fractions first).
- Degree is **not defined** if a derivative sits inside $\sin, \cos, e^{(\cdot)}, \ln, \tan^{-1}$ etc. E.g. $y'' + \sin(y') = 0$, $e^{y''} = x$.
- But $y'' + \sin y = 0$ has degree 1 — a function of y (not of a derivative) is allowed.
- $\left[1 + (y')^2\right]^{3/2} = k\,y''$: square both sides ⇒ order 2, degree 2.
- $y' = \sqrt{1 + y''}$: square ⇒ $(y')^2 = 1 + y''$: order 2, degree 1.
- $y = xy' + \sqrt{1 + (y')^2}$: isolate and square ⇒ order 1, degree 2.
- $\left(\frac{d^2y}{dx^2}\right)^{2/3} = \frac{dy}{dx}$: cube ⇒ $(y'')^2 = (y')^3$: degree 2.
- **Trap:** Reading degree straight off $(y'')^{3/2}$ as $\frac{3}{2}$. Degree is always a positive integer.
- **Trap:** Degree is of the *highest-order* derivative, not the largest power seen: in $(y')^3 + y'' = 0$ degree is 1.
- **Trap:** Order and degree are always **positive integers**; "order × degree" MCQs need both correct.

A3. FORMING A DE FROM A FAMILY

- n independent arbitrary constants ⇒ differentiate n times and eliminate ⇒ DE of order n.
- Lines $y = mx + c$: $y'' = 0$. Lines through origin $y = mx$: $xy' = y$.
- All circles in a plane: order 3. Circles through origin with centre on x-axis: order 1.
- Parabolas $y^2 = 4ax$: $2xy' = y$. Ellipses/conics with n free parameters: order n.
- $y = Ae^x + Be^{-x}$: $y'' = y$; $y = A\cos x + B\sin x$: $y'' + y = 0$.
- **Trick:** Count *independent* constants: $y = ae^{x+b} = (ae^b)e^x$ has only 1; $y = (a+b)x + c$ has 2; $y = a\sin^2 x + b\cos^2 x + c$: only 2.
- **Trap:** Over-counting merged constants gives the wrong order — a favourite JEE Main MCQ.

A4. EXACT-DIFFERENTIAL SHORTCUTS

Spot these and integrate in one line:

$x\,dy + y\,dx = d(xy)$	$\frac{x\,dy - y\,dx}{x^2} = d\left(\frac{y}{x}\right)$
$\frac{y\,dx - x\,dy}{y^2} = d\left(\frac{x}{y}\right)$	$\frac{x\,dy + y\,dx}{xy} = d(\ln xy)$
$\frac{x\,dy - y\,dx}{xy} = d\left(\ln\frac{y}{x}\right)$	$\frac{x\,dy - y\,dx}{x^2+y^2} = d\left(\tan^{-1}\frac{y}{x}\right)$
$x\,dx + y\,dy = \frac{1}{2}d(x^2+y^2)$	$\frac{x\,dx + y\,dy}{\sqrt{x^2+y^2}} = d\sqrt{x^2+y^2}$
$e^x(dy + y\,dx) = d(ye^x)$	$\frac{y\,dx + x\,dy}{(xy)^2} = -d\left(\frac{1}{xy}\right)$

- **Trick:** When $x\,dy - y\,dx$ appears, divide by x^2, y^2, xy or x^2+y^2 — whichever the rest of the equation suggests.
- E.g. $x\,dy - y\,dx = x^2\,dx \Rightarrow d\left(\frac{y}{x}\right) = dx \Rightarrow y = x^2 + Cx$.

C. WORD & GEOMETRY PATTERNS

- Tangent slope = $\frac{dy}{dx}$; normal slope = $-\frac{dx}{dy}$. Subtangent = $\left|\frac{y}{y'}\right|$, subnormal = $|yy'|$.
- **Growth/decay:** $\frac{dN}{dt} = kN \Rightarrow N = N_0e^{kt}$; half-life = $\frac{\ln 2}{|k|}$. **Cooling:** $\frac{dT}{dt} = -k(T - T_s)$.
- **Integral equation:** $y(x) = a + \int_0^x g(t, y)\,dt$ — differentiate (Leibniz) for the DE; put $x = 0$ for $y(0) = a$.
- **Functional equation:** $f(x+y) = f(x)f(y) \Rightarrow f'(x) = f'(0)f(x) \Rightarrow f = e^{kx}$.
- **Trick:** "Find $y(1)$" MCQs: solve, apply the initial condition, *then* substitute — and check the domain (e.g. $\ln x$ needs $x > 0$).

B1. VARIABLE SEPARABLE

$$\frac{dy}{dx} = f(x)\,g(y) \;\Rightarrow\; \int \frac{dy}{g(y)} = \int f(x)\,dx + C$$

- **Reducible:** $\frac{dy}{dx} = f(ax + by + c)$: put $z = ax + by + c$, $\frac{dz}{dx} = a + b\,f(z)$ ⇒ separable.
- E.g. $\frac{dy}{dx} = \cos(x + y)$: $z = x + y \Rightarrow \frac{dz}{1+\cos z} = dx \Rightarrow \tan\frac{z}{2} = x + C$.
- Factor first: $\frac{dy}{dx} = e^{x-y} + x^2e^{-y} = e^{-y}(e^x + x^2) \Rightarrow e^y = e^x + \frac{x^3}{3} + C$.
- **Trap:** Dividing by $g(y)$ silently drops solutions with $g(y) = 0$ (e.g. $y = 0$ for $y' = xy$).
- **Trap:** Put $+C$ the moment you integrate; adding it after exponentiating changes Ce^x into $e^x + C$.
- **Trap:** $\int \frac{dy}{y} = \ln|y|$; then $y = Ce^{\cdots}$ absorbs the sign — don't write $y = e^{\cdots} + C$.

B2. HOMOGENEOUS EQUATIONS

$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ with f, g homogeneous of the *same* degree, i.e. $f(\lambda x, \lambda y) = \lambda^n f(x, y) \Rightarrow \frac{dy}{dx} = F\left(\frac{y}{x}\right)$.

- Put $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx} \Rightarrow \int \frac{dv}{F(v) - v} = \int \frac{dx}{x}$. Substitute $v = \frac{y}{x}$ back at the end.
- If $\frac{dx}{dy} = G\left(\frac{x}{y}\right)$ looks simpler, use $x = vy$.
- E.g. $x\frac{dy}{dx} = y + \sqrt{x^2 + y^2}$ $(x > 0) \Rightarrow y + \sqrt{x^2 + y^2} = Cx^2$.
- **Reducible:** $\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$
  - $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$: shift $x = X + h$, $y = Y + k$ with (h, k) the intersection of the two lines ⇒ homogeneous in X, Y.
  - $\frac{a_1}{a_2} = \frac{b_1}{b_2}$: put $z = a_1x + b_1y$ ⇒ separable.
  - $b_1 = -a_2$: regroup into exact differentials ($x\,dy + y\,dx = d(xy)$) and integrate directly.
- **Trap:** Check every term's degree: $\frac{x^2+xy}{y+1}$ is *not* homogeneous; $\frac{y}{x} + \sin\frac{y}{x}$ is (degree 0).
- **Trap:** Forgetting the $x\frac{dv}{dx}$ term, or not back-substituting $v = y/x$ before using the initial condition.

B3. LINEAR FIRST-ORDER EQUATIONS

$$\frac{dy}{dx} + P(x)\,y = Q(x); \quad \text{IF} = e^{\int P\,dx}; \quad y \cdot \text{IF} = \int Q \cdot \text{IF}\,dx + C$$

- **Linear in x:** $\frac{dx}{dy} + P(y)\,x = Q(y)$; IF $= e^{\int P\,dy}$, $x \cdot \text{IF} = \int Q \cdot \text{IF}\,dy + C$.
- E.g. $y\,dx = (x + y^2)\,dy \Rightarrow \frac{dx}{dy} - \frac{x}{y} = y$: IF $= \frac{1}{y}$, $x = y^2 + Cy$.
- **IF quick table** ($e^{\ln f} = f$): $P = \frac{1}{x} \to x$; $P = -\frac{1}{x} \to \frac{1}{x}$; $P = \frac{n}{x} \to x^n$; $P = \tan x \to \sec x$; $P = -\tan x \to \cos x$; $P = \cot x \to \sin x$; $P = \frac{2x}{1+x^2} \to 1 + x^2$.
- **Bernoulli:** $\frac{dy}{dx} + Py = Qy^n$: divide by y^n, put $z = y^{1-n} \Rightarrow \frac{dz}{dx} + (1-n)Pz = (1-n)Q$.
- **Hidden linear:** $f'(y)\frac{dy}{dx} + P\,f(y) = Q$: put $z = f(y)$. E.g. $\sec^2 y\frac{dy}{dx} + P\tan y = Q$: take $z = \tan y$.
- **Trap:** Make the coefficient of $\frac{dy}{dx}$ equal to 1 *before* reading P, and keep the sign of P.
- **Trap:** No $+C$ inside the IF; the $+C$ goes after $\int Q \cdot \text{IF}$, so $y = \frac{1}{\text{IF}}\left[\int Q\,\text{IF}\,dx + C\right]$ — C gets divided too.
- **Trap:** An equation non-linear in y (e.g. y^2 terms) may be linear in x — try $\frac{dx}{dy}$ before giving up.

B4. ONE-LINE RECOGNITION GUIDE

- $\frac{dy}{dx} = \frac{y}{x} + \tan\frac{y}{x}$ → homogeneous: $\sin\frac{y}{x} = Cx$.
- $\frac{dy}{dx} + \frac{y}{x} = x^2$ → linear, IF $= x$: $xy = \frac{x^4}{4} + C$.
- $\frac{dy}{dx} = (x + y)^2$ → $z = x + y$: $\tan^{-1}(x + y) = x + C$.
- $\frac{dy}{dx} + y = y^2$ → Bernoulli ($n = 2$), or separable directly.
- $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$ → linear in x, IF $= e^{\tan^{-1} y}$.

EXAM-DAY CHECKLIST

- Classify in this order: separable → reducible by $z = ax + by + c$ → exact grouping → homogeneous → linear in y → linear in x → Bernoulli.
- Write $+C$ once, at integration; find it from the condition before simplifying.
- Verify the final answer by differentiating it, or by checking the given point.

For personal revision only