INTEGRAL CALCULUS: INDEFINITE & DEFINITE — TRICKS, TRAPS & AREA UNDER CURVES

A1. CORE ANTIDERIVATIVES (memorise cold)

$\int x^n dx = \frac{x^{n+1}}{n+1} (n \neq -1)$; $\int \frac{dx}{x} = \ln|x|$

$\int a^x dx = \frac{a^x}{\ln a}$; $\int \ln x\, dx = x\ln x - x$

$\int \tan x\, dx = \ln|\sec x|$; $\int \cot x\, dx = \ln|\sin x|$

$\int \sec x\, dx = \ln|\sec x + \tan x| = \ln\left|\tan\left(\frac{x}{2}+\frac{\pi}{4}\right)\right|$

$\int \csc x\, dx = \ln\left|\tan\frac{x}{2}\right| = -\ln|\csc x + \cot x|$

$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$; $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right|$

$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$; $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$

$\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln\left|x+\sqrt{x^2\pm a^2}\right|$

$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right|$

$\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}$

$\int \sqrt{x^2\pm a^2}\,dx = \frac{x}{2}\sqrt{x^2\pm a^2} \pm \frac{a^2}{2}\ln\left|x+\sqrt{x^2\pm a^2}\right|$

- **Trap:** Always $+C$; $\ln|x|$ with modulus. Marks are cut for $\ln x$.
- $\int f(ax+b)dx = \frac{1}{a}F(ax+b)$ — divide by a, students forget.
- **Trap:** $\int \frac{dx}{x^2+a^2}$ vs $\int \frac{dx}{\sqrt{a^2-x^2}}$: $\tan^{-1}$ has $\frac{1}{a}$, $\sin^{-1}$ has none.

A2. SUBSTITUTION – PATTERN TABLE

- $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)|$; $\int f'(x)[f(x)]^n dx = \frac{[f]^{n+1}}{n+1}$.
- See $\sqrt{a^2-x^2} \Rightarrow x = a\sin\theta$; $\sqrt{a^2+x^2} \Rightarrow x = a\tan\theta$; $\sqrt{x^2-a^2} \Rightarrow x = a\sec\theta$.
- $\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{x-\alpha}{\beta-x}} \Rightarrow x = a\cos 2\theta$ / $x = \alpha\cos^2\theta + \beta\sin^2\theta$.
- $\frac{1}{(x-\alpha)^k}$ type, or $\frac{dx}{(x-\alpha)\sqrt{ax^2+bx+c}} \Rightarrow x-\alpha = \frac{1}{t}$.
- $\frac{dx}{(ax^2+b)\sqrt{cx^2+d}} \Rightarrow x = \frac{1}{t}$, then substitute for the surd.
- Only even powers of x in a hard rational $\Rightarrow$ try $x^2 = t$.
- $\int \frac{dx}{x(x^n+1)}$: multiply by $\frac{x^{n-1}}{x^{n-1}}$, put $x^n+1 = t$.

Trick: $\int \frac{x^2\pm1}{x^4+kx^2+1}dx$: divide num & den by x^2, write as $1\pm\frac{1}{x^2}$, use $t = x\mp\frac{1}{x}$ with $x^2+\frac{1}{x^2} = t^2\pm2$. Splitting $\frac{2x^2}{x^4+1}$ into $(x^2+1)+(x^2-1)$ cracks the "+1 and −1" pair.

- **Trap:** Changing variable but forgetting to change dx, or re-substituting back in an indefinite answer.

A3. INTEGRATION BY PARTS

$\int u\,v\,dx = u\int v\,dx - \int\left(u'\int v\,dx\right)dx$, order by **ILATE** (Inverse, Log, Algebraic, Trig, Exponential) for choosing u.

- $\int \ln x\,dx$, $\int \sin^{-1}x\,dx$, $\int \tan^{-1}x\,dx$: take $v = 1$.
- **Golden result:** $\int e^x\{f(x)+f'(x)\}dx = e^x f(x) + C$. Spot $f + f'$ after factoring e^x. e.g. $\int e^x\frac{1+x\ln x}{x}dx = e^x\ln x$.
- $\int e^{ax}\sin bx\,dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2}$ (swap for cos: $a\cos bx + b\sin bx$).
- $\int\{xf'(x)+f(x)\}dx = xf(x)$; $\int e^x\frac{(x-1)}{(x+1)^3}dx$ type: force $f + f'$ form.
- Loop trick: if the same integral I reappears, solve $I = \cdots - kI$.
- **Trap:** Wrong ILATE order flips a 2-line sum into an endless loop. $\int \sin^{-1}x\cdot x\,dx$: inverse is u, not x.

A4. PARTIAL FRACTIONS

- Make **proper** first (divide if $\deg N \geq \deg D$) — most common silly error.
- $(x-a) \to \frac{A}{x-a}$; $(x-a)^2 \to \frac{A}{x-a} + \frac{B}{(x-a)^2}$; irreducible $(x^2+bx+c) \to \frac{Ax+B}{x^2+bx+c}$.
- Cover-up: $A = \left.\frac{N(x)}{D'(x)}\right|_{x=a}$ for distinct linear roots — fastest in MCQs.
- $\frac{1}{(x-a)(x-b)} = \frac{1}{a-b}\left[\frac{1}{x-a} - \frac{1}{x-b}\right]$ (telescoping friend).
- For $\frac{P(x)}{x^n(\dots)}$ substituting $x = \frac{1}{t}$ often beats full decomposition.

A5. QUADRATIC-DENOMINATOR TOOLKIT

Always **complete the square**: $ax^2+bx+c = a\left[\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right]$, then match a core form.

- $\int \frac{px+q}{ax^2+bx+c}dx$: write $px+q = \lambda(2ax+b)+\mu$. Then $\lambda\ln|\cdot| + \mu\int\frac{dx}{ax^2+bx+c}$.
- Same split works for $\int \frac{px+q}{\sqrt{ax^2+bx+c}}dx$ $(= 2\lambda\sqrt{Q} + \mu\int\frac{dx}{\sqrt{Q}}$, $Q = ax^2+bx+c)$ and for $\int(px+q)\sqrt{ax^2+bx+c}\,dx$.
- $\int \frac{dx}{a\sin x + b\cos x}$: write $= \frac{1}{r}\csc(x+\phi)$ with $r = \sqrt{a^2+b^2}$.
- **Trap:** Sign of $4ac-b^2$ decides $\tan^{-1}$ vs ln vs $\sin^{-1}$ — check the discriminant before quoting a formula.
- **Trap:** $\sqrt{a^2} = |a|$ when you pull terms out of a square root after completing the square.

A6. TRIGONOMETRIC INTEGRALS

- Odd power of sin $\Rightarrow$ $t = \cos x$ (and vice-versa); both even $\Rightarrow$ power-reduction $\sin^2 = \frac{1-\cos 2x}{2}$.
- $\int \frac{dx}{a+b\cos x}$, $\frac{dx}{a+b\sin x}$: $t = \tan\frac{x}{2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2dt}{1+t^2}$.
- $\int \frac{dx}{a\sin^2 x + b\cos^2 x}$ (all terms degree 2): divide by $\cos^2 x$, put $t = \tan x$.
- $\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}dx$: write Num $= \lambda$(Den) $+ \mu$(Den)$' + \nu$.
- $\sqrt{1\pm\sin x} = |\sin\frac{x}{2} \pm \cos\frac{x}{2}|$ — **Trap:** dropping modulus changes sign on half the domain.
- Reduction: $I_n = \int\sin^n x\,dx = -\frac{\sin^{n-1}x\cos x}{n} + \frac{n-1}{n}I_{n-2}$.

B1. FUNDAMENTAL THEOREM & LEIBNIZ

$\int_a^b f(x)dx = F(b) - F(a)$ (valid only if f is continuous on $[a,b]$).

$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h)h'(x) - f(g)g'(x)$ (Newton–Leibniz).

- Definite integral is a **number**: no $+C$, variable of integration is dummy: $\int_a^b f(x)dx = \int_a^b f(t)dt$.
- **Trap:** Applying FTC across a discontinuity. $\int_{-1}^{1}\frac{dx}{x^2} \neq -2$ (it diverges); check the integrand blows up inside.
- **Trap:** $\int_0^\pi \frac{dx}{1+2\sin^2 x}$ with $t = \tan x$: tan is undefined at $\frac{\pi}{2}$. Split at $\frac{\pi}{2}$ and use symmetry.
- If $\int_a^b f = 0$ and f continuous, f must vanish somewhere in (a,b).

B2. PROPERTY TOOLKIT (highest yield)

- **King:** $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$. Add $I+I$ when $f(x)+f(a+b-x)$ is constant. Classic: $\int_0^{\pi/2}\frac{\sin^n x}{\sin^n x+\cos^n x}dx = \frac{\pi}{4}$ $\forall n$.
- **Even/Odd:** $\int_{-a}^a f = 2\int_0^a f$ if even, 0 if odd. Split f into even+odd parts to kill half the work.
- $\int_0^{2a} f(x)dx = \int_0^a [f(x)+f(2a-x)]dx$; $= 2\int_0^a f$ if $f(2a-x) = f(x)$, $= 0$ if $f(2a-x) = -f(x)$.
- **Periodic T:** $\int_0^{nT} f = n\int_0^T f$; $\int_a^{a+T} f = \int_0^T f$ for every a.
- $\int_0^{\pi/2} f(\sin x)dx = \int_0^{\pi/2} f(\cos x)dx$. Also $\int_0^\pi x\,f(\sin x)dx = \frac{\pi}{2}\int_0^\pi f(\sin x)dx$ — kills the x.
- **Walli's:** $\int_0^{\pi/2}\sin^m x\cos^n x\,dx = \frac{[(m-1)(m-3)\cdots][(n-1)(n-3)\cdots]}{(m+n)(m+n-2)\cdots}\cdot k$, where $k = \frac{\pi}{2}$ if m,n are *both* even, else $k = 1$; e.g. $\int_0^{\pi/2}\sin^4 x\,dx = \frac{3\cdot1}{4\cdot2}\cdot\frac{\pi}{2}$.
- $|\int f| \leq \int|f|$; if $m \leq f \leq M$ then $m(b-a) \leq \int_a^b f \leq M(b-a)$ (used for "which interval" MCQs).
- **Trap:** Using even/odd when the integrand has $|x|, [x], \{x\}$ or sgn — recheck parity; $[x]$ is *not* odd.
- **Trap:** $\int_a^b f$ with $a > b$: sign flips. Never ignore limit order after substitution — always convert limits, never back-substitute.

B3. SPECIAL EVALUATIONS

- $\int_0^{\pi/2}\ln(\sin x)dx = \int_0^{\pi/2}\ln(\cos x)dx = -\frac{\pi}{2}\ln 2$.
- $\int_0^{\pi/2}\frac{dx}{1+\tan^n x} = \frac{\pi}{4}$; $\int_0^1\frac{\ln(1+x)}{1+x^2}dx = \frac{\pi}{8}\ln 2$.
- **Modulus/GIF:** split the interval at zeros/integer points: $\int_0^3|x-1|dx$, $\int_0^n[x]dx = \frac{n(n-1)}{2}$, $\int_0^n\{x\}dx = \frac{n}{2}$.
- **Limit of a sum:** $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)dx$. Replace $\frac{r}{n} \to x$, $\frac{1}{n} \to dx$; limits $= \lim\frac{r}{n}$ at $r = 1$ and $r = n$.
- **Trap:** In limit-of-sum, wrong upper limit (e.g. r up to $2n$ gives $\int_0^2$). Check both ends.

C. AREA UNDER CURVES

- Area $= \int_a^b|y|dx$ (strip $\parallel$ y-axis) or $\int_c^d|x|dy$ (strip $\parallel$ x-axis). Choose whichever needs *fewer* splits.
- Between curves: $\int_a^b(y_{top}-y_{bot})dx$ with a,b the intersection abscissae.
- **Trap:** Region below the x-axis gives a negative integral. Split at every zero of f and add absolute values — area is never negative.
- Parabola–chord (Archimedes): area cut from $y = ax^2+bx+c$ by a line meeting it at $x = \alpha,\beta$ is $\frac{|a|(\beta-\alpha)^3}{6} = \frac{(b^2-4ac)^{3/2}}{6a^2}$ for the x-axis chord.
- Ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$: area $= \pi ab$; circle πr^2; parabola $y^2 = 4ax$ & latus rectum: $\frac{8a^2}{3}$.
- $y^2 = 4ax$ and $x^2 = 4by$ enclose $\frac{16ab}{3}$; $y^2 = 4ax$ inside $x^2+y^2 = a^2$ etc. — always sketch first.
- Symmetry saves time: even function or quadrant symmetry $\Rightarrow$ compute one part, multiply.
- **Trap:** Forgetting that $\sqrt{\cdot}$ gives only the upper branch; $y^2 = 4ax$ needs $2\int y\,dx$ for the full region.

D. RESULTS THAT KEEP REPEATING

- $\int\sec^3 x\,dx = \frac{1}{2}[\sec x\tan x + \ln|\sec x+\tan x|]$.
- $\int\frac{dx}{\sin x\cos x} = \ln|\tan x|$; $\int\frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^2}\left[\frac{x}{x^2+a^2}+\frac{1}{a}\tan^{-1}\frac{x}{a}\right]$.
- $\int\frac{xe^x}{(1+x)^2}dx = \frac{e^x}{1+x}$; $\int e^x\left(\frac{1}{x}-\frac{1}{x^2}\right)dx = \frac{e^x}{x}$ $(f+f'$ pattern).
- $\int\frac{dx}{1+x^4}$: divide by x^2, split into $t = x\mp\frac{1}{x}$ pair.
- $\int_0^1 x^m(1-x)^n dx = \frac{m!\,n!}{(m+n+1)!}$ (integers $m,n \geq 0$).
- $\int_0^{\pi/2}\sqrt{\tan x}\,dx = \frac{\pi}{\sqrt{2}}$; $\int_0^\pi|\cos x|dx = 2$, $\int_0^{n\pi}|\sin x|dx = 2n$.
- $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ — the one-line version of King, used in most property questions.
- $\int\tan^{-1}x\,dx = x\tan^{-1}x - \frac{1}{2}\ln(1+x^2)$; $\int\sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2}$.

EXAM-DAY CHECKLIST

- Differentiate the options — in indefinite-integral MCQs, back-differentiation is faster than integrating.
- Verify an answer by plugging a convenient value (e.g. $x = 0, 1$) or comparing numerically.
- Two answers may differ by a constant and both be "correct" — match the form given in options.
- Domain check: ln needs positive argument, $\sin^{-1}$ needs $|x| \leq 1$, $\sec^{-1}$ needs $|x| \geq 1$.
- If an integral looks impossible (e^{-x^2}, $\frac{\sin x}{x}$, $\frac{1}{\ln x}$), the question wants a *property*, not evaluation.

For personal revision only